

Lipschitz equivalence of a class of self-similar sets

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Abstract

We consider a class of homogeneous self-similar sets with complete overlaps and give a sufficient condition for the Lipschitz equivalence between members in this class.

Keywords: complete overlap; homogeneous self-similar sets; Lipschitz equivalence

AMS Subject Classifications: 28A80, 28A78.

1 Introduction

Let $(X_i, d_i), i = 1, 2$ be metric spaces. For nonempty sets $A_i \subseteq X_i$ we say they are Lipschitz equivalent, denoted by $A_1 \simeq A_2$, if there exists a bijection $\phi : A_1 \rightarrow A_2$ and a constant $c > 0$ such that

$$c^{-1}d_1(x, y) \leq d_2(\phi(x), \phi(y)) \leq cd_1(x, y) \text{ for any } x, y \in A_1.$$

Lipschitz equivalence can be used to classify fractal sets. Since late 80's many works have been devoted to the study of Lipschitz equivalence (see [2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 14] and references therein). An effective method, to our knowledge, was first employed in [10] for establishing a bi-Lipschitz mapping between the $\{1, 4, 5\}$ -Cantor set and the $\{1, 3, 5\}$ -Cantor set, the main idea of which is to show these two self-similar sets to have same graph-directed structure satisfying the strong separation condition. A sufficient condition was given in [1, Theorem 2.11] to judge whether or not a self-similar set has a graph-directed structure satisfying the open set condition or even the strong separation condition.

In the present we consider the homogeneous iterated function system (IFS) $\{f_i(x) = \lambda x + a_i : 1 \leq i \leq m\}$ where $x, a_i \in \mathbb{R}$, $\lambda \in (0, 1)$ and the integer $m \geq 3$. For a vector (k_1, \dots, k_n) of integers with $k_1 > k_2 > \dots > k_n \geq 2$, let $\mathbb{A}_{k_1, \dots, k_n}$ be the collection of translations $\mathbf{a} = (a_1, a_2, \dots, a_m)$ satisfying the following conditions (I) (II) and (III):

(I) $0 = a_1 < a_2 < \dots < a_m = 1 - \lambda$;

(II) Any three intervals in $\{f_i([0, 1]) : 1 \leq i \leq m\}$ do not intersect. $|f_i([0, 1]) \cap f_j([0, 1])| \in \{\lambda^{k_1}, \dots, \lambda^{k_n}\}$ whenever $f_i([0, 1]) \cap f_j([0, 1]) \neq \emptyset$ with $i < j$, where by $|J|$ we denote the length of an interval J ;

(III) Either $f_1([0, 1]) \cap f_j([0, 1]) = \emptyset$ for all $j > 1$, or $f_m([0, 1]) \cap f_j([0, 1]) = \emptyset$ for all $j < m$.

From (I) and (II) it follows that when $|f_i([0, 1]) \cap f_j([0, 1])| = \lambda^{k_\ell}$ with $i < j$, then $j = i + 1$ and $f_i \circ f_m^{k_\ell - 1} = f_j \circ f_1^{k_\ell - 1}$. Throughout this paper, f^i stands for the i -th iteration of map f for $i \in \mathbb{N} \cup \{0\}$. In particular, f^0 stands for the identity.

For a translation $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{A}_{k_1, \dots, k_n}$, Let

$$\gamma_\ell(\mathbf{a}) = \{1 \leq i \leq m : |f_i([0, 1]) \cap f_{i+1}([0, 1])| = \lambda^{k_\ell}\} \quad \text{for } 1 \leq \ell \leq n.$$

It is well known that for each $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{A}_{k_1, \dots, k_n}$, there exists a unique nonempty compact set $K_{\mathbf{a}}$ such that $K_{\mathbf{a}} = \bigcup_{1 \leq i \leq m} f_i(K_{\mathbf{a}})$ (see [5]). The set $K_{\mathbf{a}}$ is called the self-similar set generated by the IFS $\{f_i(x) = \lambda x + a_i : 1 \leq i \leq m\}$. Let $\#S$ denote the number of elements of S . In this paper we obtain

Theorem 1.1. *For $\mathbf{a}, \mathbf{b} \in \mathbb{A}_{k_1, \dots, k_n}$ we have $K_{\mathbf{a}} \simeq K_{\mathbf{b}}$ if $\#\gamma_\ell(\mathbf{a}) = \#\gamma_\ell(\mathbf{b})$ for $1 \leq \ell \leq n$.*

It is clear that $\dim_H E = \dim_H F$ if $E \simeq F$. Thus we have

Corollary 1.2. *For $\mathbf{a}, \mathbf{b} \in \mathbb{A}_{k_1, \dots, k_n}$ we have $\dim_H K_{\mathbf{a}} = \dim_H K_{\mathbf{b}}$ if $\#\gamma_\ell(\mathbf{a}) = \#\gamma_\ell(\mathbf{b})$ for $1 \leq \ell \leq n$.*

We prove Theorem 1.1 and give some examples in the next section.

2 Proof of Theorem 1.1

Before proving Theorem 1.1, let us recall the graph-directed self-similar set (see [9]). Let $\mathcal{G} = (V, E)$ be a directed graph where V is a finite set of vertexes and E is a finite set of directed edges. Assume that for any $u \in V$ there is at least one edge in E starting from u . For an $e \in E$, let $f_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a similitude with ratio $\rho_e \in (0, 1)$, namely

$$|f_e(x) - f_e(y)| = \rho_e |x - y| \quad \text{for any } x, y \in \mathbb{R}^n.$$

Then there exist unique nonempty compact sets $\{F_u : u \in V\}$ such that

$$F_u = \bigcup_{v \in V} \bigcup_{e \in E_{u,v}} f_e(F_v) \quad \text{for all } u \in V, \quad (1)$$

where $E_{u,v}$ is the set of directed edges starting from u and ending at v . The compact sets $\{F_u : u \in V\}$ in (1) is called the graph-directed self-similar sets generated by $\{V, E, \{f_e : e \in E\}\}$. In addition, $\{F_u : u \in V\}$ is said to satisfy the strong separation condition if the sets in the right side of (1) are pairwise disjoint. An easy-to-prove result on the Lipschitz equivalence between two graph-directed self-similar sets is as follows (also see [10]).

Lemma 2.1. *Let $\{F_u : u \in V\}$ and $\{G_u : u \in V\}$ be the graph-directed self-similar sets generated by $\{V, E, \{f_e : e \in E\}\}$ and $\{V, E, \{g_e : e \in E\}\}$, respectively. Suppose that for each $e \in E$ the similitudes f_e and g_e have the same ratio ρ_e , and both $\{F_u : u \in V\}$ and $\{G_u : u \in V\}$ satisfy the strong separation condition. Then for each $u \in V$, we have $F_u \simeq G_u$.*

Proof. Fix a $u \in V$. We denote by E_v the set of directed edges starting from v for $v \in V$. For a directed edge $e \in E$ we denote its initial and ending points by e^- and e^+ , respectively. Let

$$c_* = \min_{v \in V} \min \left\{ d(f_{e_*}(F_{e_*^+}), f_{e_{**}}(F_{e_{**}^+})), d(g_{e_*}(G_{e_*^+}), g_{e_{**}}(G_{e_{**}^+})) : e_* \neq e_{**} \in E_v \right\}$$

and

$$c^* = \max \left\{ \text{diameter of the set } \bigcup_{v \in V} F_v, \text{ diameter of the set } \bigcup_{v \in V} G_v \right\}.$$

Then $c_*, c^* > 0$. An infinite sequence of directed edges $e_1 e_2 \dots$ is called admissible if e_i^+ coincides with e_{i+1}^- for all $i \in \mathbb{N}$. Let

$$\Sigma_u = \{e_1 e_2 \dots : e_1 e_2 \dots \text{ is admissible with } e_1^- = u\}.$$

Then the maps

$$\Pi_F(e_1 e_2 \dots) = \bigcap_{i=1}^{\infty} f_{e_1} \circ \dots \circ f_{e_i}(F_{e_i^+}) \text{ and } \Pi_G(e_1 e_2 \dots) = \bigcap_{i=1}^{\infty} g_{e_1} \circ \dots \circ g_{e_i}(G_{e_i^+})$$

are bijections between Σ_u and F_u , and between Σ_u and G_u respectively. We shall check the bijection $\Pi_G \circ \Pi_F^{-1}$ is bi-Lipschitz. Let $x, y \in F_u$ with $x \neq y$. Then there exist unique $(e_i), (s_i) \in \Sigma_u$ such that $x = \Pi_F(e_1 e_2 \dots), y = \Pi_F(s_1 s_2 \dots)$. Let ℓ be the smallest integer such that $e_\ell \neq s_\ell$. Then we have $e_\ell^- = s_\ell^-$ and $e_\ell^+ \neq s_\ell^+$ because of $e_1^- = s_1^- = u$. This implies that $x = f_{e_1} \circ \dots \circ f_{e_{\ell-1}}(x^*)$ and $y = f_{e_1} \circ \dots \circ f_{e_{\ell-1}}(y^*)$ with $x^* \in f_{e_\ell}(F_{e_\ell^+}), y^* \in f_{s_\ell}(F_{s_\ell^+})$. So

$$c_* \prod_{i=1}^{\ell-1} \rho_{e_i} \leq |x - y| \leq c^* \prod_{i=1}^{\ell-1} \rho_{e_i},$$

Note that

$$\Pi_G \circ \Pi_F^{-1}(x) = \Pi_G(e_1 e_2 \dots) \text{ and } \Pi_G \circ \Pi_F^{-1}(y) = \Pi_G(s_1 s_2 \dots),$$

which implies, by the same argument as above, that

$$c_* \prod_{i=1}^{\ell-1} \rho_{e_i} \leq |\Pi_G \circ \Pi_F^{-1}(x) - \Pi_G \circ \Pi_F^{-1}(y)| \leq c^* \prod_{i=1}^{\ell-1} \rho_{e_i}.$$

Therefore, $\Pi_G \circ \Pi_F^{-1}$ is bi-Lipschitz. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By $\{f_i : 1 \leq i \leq m\}$ and $\{g_i : 1 \leq i \leq m\}$ we denote the iterated function systems corresponding to translations $\mathbf{a} = (a_1, a_2, \dots, a_m)$ and $\mathbf{b} = (b_1, b_2, \dots, b_m)$, respectively.

Without loss of generality we may assume that $f_m([0, 1]) \cap f_j([0, 1]) = \emptyset$ for all $j < m$, and that $g_m([0, 1]) \cap g_j([0, 1]) = \emptyset$ for all $j < m$ in condition (III). To understand it,

one only need to notice the following facts: we have that $K_{\mathbf{c}} = 1 - K_{\mathbf{a}} \simeq K_{\mathbf{a}}$ for the translation $\mathbf{c} = (1 - \lambda - a_m, 1 - \lambda - a_{m-1}, \dots, 1 - \lambda - a_2, 1 - \lambda - a_1) \in \mathbb{A}_{k_1, \dots, k_n}$, and $\#\gamma_\ell(\mathbf{c}) = \#\gamma_\ell(\mathbf{a})$ for $1 \leq \ell \leq n$. Let

$$\gamma_{n+1}(\mathbf{a}) = \{1, \dots, m-1\} \setminus \bigcup_{\ell=1}^n \gamma_\ell(\mathbf{a}).$$

We relabel the elements of $\gamma_\ell(\mathbf{a})$ in its increasing order by digits $\{1 + \sum_{j=1}^{\ell-1} \#\gamma_j(\mathbf{a}), 2 + \sum_{j=1}^{\ell-1} \#\gamma_j(\mathbf{a}), \dots, \sum_{j=1}^{\ell} \#\gamma_j(\mathbf{a})\}$ with $\sum_{j=1}^0 \#\gamma_j(\mathbf{a}) = 0$ and $1 \leq \ell \leq n+1$. By $h(\cdot)$ we denote this relabeling. Thus $h(j) \in \{1 + \sum_{j=1}^{\ell-1} \#\gamma_j(\mathbf{a}), 2 + \sum_{j=1}^{\ell-1} \#\gamma_j(\mathbf{a}), \dots, \sum_{j=1}^{\ell} \#\gamma_j(\mathbf{a})\}$ for $j \in \gamma_\ell(\mathbf{a})$.

We partition $K_{\mathbf{a}}$ into $m + k_1 - 2$ pairwise disjoint nonempty compact sets, denoted by K_1, \dots, K_{m+k_1-2} . The first $m-1$ members of them are defined by

$$K_{h(j)} = \begin{cases} f_j(K_{\mathbf{a}}) \setminus f_j \circ f_m^{k_\ell-1}(K_{\mathbf{a}}) & \text{for } j \in \gamma_\ell(\mathbf{a}), 1 \leq \ell \leq n \\ f_j(K_{\mathbf{a}}); & \text{for } j \in \gamma_{n+1}(\mathbf{a}). \end{cases} \quad (2)$$

The later $k_1 - 1$ members of K_i s are defined by

$$\begin{cases} K_{m+k_1-2} = f_m^{k_1-1}(K_{\mathbf{a}}) \\ K_{m+t} = f_m^{t+1}(K_{\mathbf{a}}) \setminus f_m^{t+2}(K_{\mathbf{a}}) \text{ for } 0 \leq t < k_1 - 2. \end{cases} \quad (3)$$

From (2) and (3) it follows that $K_{\mathbf{a}} = \bigcup_{i=1}^{m+k_1-2} K_i$ with disjoint union. It is important to notice that for $1 \leq \ell \leq n$

$$f_m(K_{\mathbf{a}}) = \begin{cases} K_m \cup K_{m+1} \cup \dots \cup K_{m+k_\ell-3} \cup f_m^{k_\ell-1}(K_{\mathbf{a}}) & \text{with disjoint union for } k_\ell \geq 3 \\ f_m(K_{\mathbf{a}}) & \text{for } k_\ell = 2. \end{cases}$$

Thus we have for $j \in \gamma_\ell(\mathbf{a})$ with $1 \leq \ell \leq n$

$$\begin{aligned} K_{h(j)} &= f_j(K_{\mathbf{a}}) \setminus f_j \circ f_m^{k_\ell-1}(K_{\mathbf{a}}) = (f_j(K_1 \cup \dots \cup K_{m-1} \cup f_m(K_{\mathbf{a}}))) \setminus f_j \circ f_m^{k_\ell-1}(K_{\mathbf{a}}) \\ &= \bigcup_{i=1}^{m+k_\ell-3} f_j(K_i). \end{aligned}$$

It is obvious that for $j \in \gamma_{n+1}(\mathbf{a})$

$$K_{h(j)} = \bigcup_{i=1}^{m+k_1-2} f_j(K_i).$$

Finally, Note that $K_{\mathbf{a}} = K_1 \cup \dots \cup K_{m-1} \cup f_m(K_{\mathbf{a}})$ with disjoint union. Thus, for $0 \leq t < k_1 - 2$

$$K_{m+t} = f_m^{t+1}(K_{\mathbf{a}}) \setminus f_m^{t+2}(K_{\mathbf{a}}) = \bigcup_{i=1}^{m-1} f_m^{t+1}(K_i)$$

and

$$K_{m+k_1-2} = \begin{cases} f_m(K_{m+k_1-3}) \cup f_m(K_{m+k_1-2}) & \text{when } k_1 \geq 3 \\ \bigcup_{i=1}^m f_m(K_i) & \text{when } k_1 = 2. \end{cases}$$

Therefore, $(K_1, \dots, K_{m+k_1-2})$ are graph-directed self-similar sets satisfying the strong separation condition. By the same argument as above by replacing f_i by g_i , one

can get pairwise disjoint nonempty compacts K_i^* with $K_{\mathbf{b}} = \bigcup_{1 \leq i \leq m+k_1-2} K_i^*$. The $(K_1^*, \dots, K_{m+k_1-2}^*)$ are graph-directed self-similar sets satisfying the strong separation condition and obey the same equations as K_i s with replacing f_i by g_i . Thus $K_i \simeq K_i^*$ for $1 \leq i \leq m+k_1-2$, and so $K_{\mathbf{a}} \simeq K_{\mathbf{b}}$ because of the disjointness of K_i s and disjointness of K_i^* s. \square

Example 2.2. Let $0 < \lambda < 5^{-1}$. Take $\mathbf{a} = (0, \lambda(1-\lambda), 2\lambda(1-\lambda), 3\lambda, 1-\lambda)$ and $\mathbf{b} = (0, \lambda(1-\lambda), 2\lambda, 3\lambda-\lambda^2, 1-\lambda)$. Then one can check that $\mathbf{a}, \mathbf{b} \in \mathbb{A}_2$, $\gamma_1(\mathbf{a}) = \{1, 2\}$ and $\gamma_1(\mathbf{b}) = \{1, 3\}$. Thus $K_{\mathbf{a}} \simeq K_{\mathbf{b}}$ by Theorem 1.1.

The approach presented in this paper can be also applied for higher dimensional case.

Example 2.3. Let $0 < \lambda < (2 - \sqrt{2})/2$. Consider two IFSs $\{f_i : 1 \leq i \leq 6\}$ and $\{g_i : 1 \leq i \leq 6\}$ where

$$\begin{aligned} f_1(x, y) &= \lambda(x, y), & f_2(x, y) &= \lambda(x, y) + (1-\lambda, 0), \\ f_3(x, y) &= \lambda(x, y) + (1-\lambda, 1-\lambda), & f_4(x, y) &= \lambda(x, y) + (0, 1-\lambda), \\ f_5(x, y) &= \lambda(x, y) + (\lambda(1-\lambda), (1-\lambda)^2), & f_6(x, y) &= \lambda(x, y) + (0, (1-\lambda)(1-2\lambda)), \end{aligned}$$

and $g_6(x, y) = \lambda(x, y) + (\lambda(1-\lambda), \lambda(1-\lambda))$ with $g_i(x, y) = f_i(x, y)$ for $1 \leq i \leq 5$. Let F and G be the self-similar sets generated by IFSs $\{f_i : 1 \leq i \leq 6\}$ and $\{g_i : 1 \leq i \leq 6\}$, respectively. Then $F \simeq G$.

Proof. Figure 1 shows locations of squares $f_i([0, 1]^2)$, $1 \leq i \leq 6$ and squares $g_i([0, 1]^2)$, $1 \leq i \leq 6$. Let $F_i = f_i(F)$ for $i = 1, 2, 3, 5$, $F_4 = f_4(F) \setminus f_4 \circ f_2(F)$ and $F_6 = f_6(F) \setminus f_6 \circ f_3(F)$. Then F_i , $1 \leq i \leq 6$, are pairwise disjoint nonempty compact sets such that $F = \bigcup_{1 \leq i \leq 6} F_i$ since $f_4 \circ f_2 = f_5 \circ f_4$ and $f_6 \circ f_3 = f_5 \circ f_1$.

Thus we have

$$\begin{cases} F_i = f_i(F_1) \cup f_i(F_2) \cup f_i(F_3) \cup f_i(F_4) \cup f_i(F_5) \cup f_i(F_6) & \text{for } i = 1, 2, 3, 5 \\ F_4 = f_4(F_1) \cup f_4(F_3) \cup f_4(F_4) \cup f_4(F_5) \cup f_4(F_6) \\ F_6 = f_6(F_1) \cup f_6(F_2) \cup f_6(F_4) \cup f_6(F_5) \cup f_6(F_6) \end{cases}$$

By the same way as above let $G_1 = g_5(G)$, $G_2 = g_2(G)$, $G_3 = g_3(G)$, $G_4 = g_4(G) \setminus g_4 \circ g_2(G)$, $G_5 = g_6(G)$ and $G_6 = g_1(G) \setminus g_1 \circ g_3(G)$. We have G_i , $1 \leq i \leq 6$, are pairwise disjoint nonempty compact sets with $G = \bigcup_{1 \leq i \leq 6} G_i$ and satisfy

$$\begin{cases} G_1 = g_5(G_1) \cup g_5(G_2) \cup g_5(G_3) \cup g_5(G_4) \cup g_5(G_5) \cup g_5(G_6) \\ G_2 = g_2(G_1) \cup g_2(G_2) \cup g_2(G_3) \cup g_2(G_4) \cup g_2(G_5) \cup g_2(G_6) \\ G_3 = g_3(G_1) \cup g_3(G_2) \cup g_3(G_3) \cup g_3(G_4) \cup g_3(G_5) \cup g_3(G_6) \\ G_5 = g_6(G_1) \cup g_6(G_2) \cup g_6(G_3) \cup g_6(G_4) \cup g_6(G_5) \cup g_6(G_6) \\ G_4 = g_4(G_1) \cup g_4(G_3) \cup g_4(G_4) \cup g_4(G_5) \cup g_4(G_6) \\ G_6 = g_1(G_1) \cup g_1(G_2) \cup g_1(G_4) \cup g_1(G_5) \cup g_1(G_6). \end{cases}$$

Thus $F \simeq G$ by Lemma 2.1. \square

Example 2.4. Let $0 < \lambda < \frac{1}{7}$. Let G be the self-similar set generated by the IFS $\{g_i : 1 \leq i \leq 6\}$ given in Example 2.3. Let F be the self-similar set generated by the IFS $\{f_i : 1 \leq i \leq 6\}$ where $f_1(x) = \lambda x$, $f_2(x) = \lambda x + 2\lambda$, $f_3(x) = \lambda x + 3\lambda - \lambda^2$, $f_4(x) = \lambda x + 4\lambda - 2\lambda^2$, $f_5(x) = \lambda x + 5\lambda$ and $f_6(x) = \lambda x + 1 - \lambda$. Then $F \simeq G$.

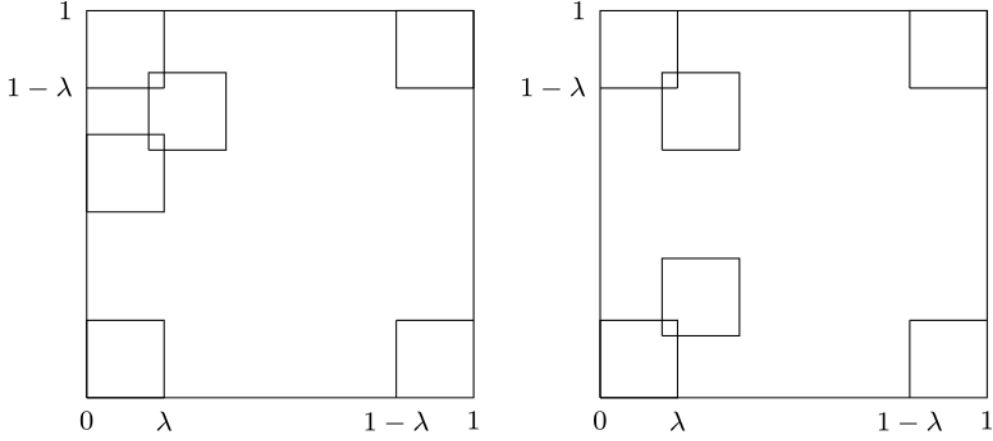


Figure 1: Squares $f_i([0, 1]^2)$ on the left side and Squares $g_i([0, 1]^2)$ on the right side

Proof. Note that $F^* = F \times \{0\}$ is the self-similar set in \mathbb{R}^2 generated by the IFS:

$$\begin{aligned} f_1^*(x, y) &= \lambda(x, y), & f_2^*(x, y) &= \lambda(x, y) + (2\lambda, 0), \\ f_3^*(x, y) &= \lambda(x, y) + (3\lambda - \lambda^2, 0), & f_4^*(x, y) &= \lambda(x, y) + (4\lambda - 2\lambda^2, 0), \\ f_5^*(x, y) &= \lambda(x, y) + (5\lambda, 0), & f_6^*(x, y) &= \lambda(x, y) + (1 - \lambda, 0). \end{aligned}$$

By letting $F_1 = f_3^*(F^*)$, $F_2 = f_1^*(F^*)$, $F_3 = f_6^*(F^*)$, $F_5 = f_5^*(F^*)$, $F_6 = f_2^*(F^*) \setminus f_2^* \circ f_6^*(F^*)$, $F_4 = f_4^*(F^*) \setminus f_4^* \circ f_1^*(F^*)$, one can get F^* has the same graph-directed structure as G . Thus we have $G \simeq F^* \simeq F$ by Lemma 2.1. \square

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